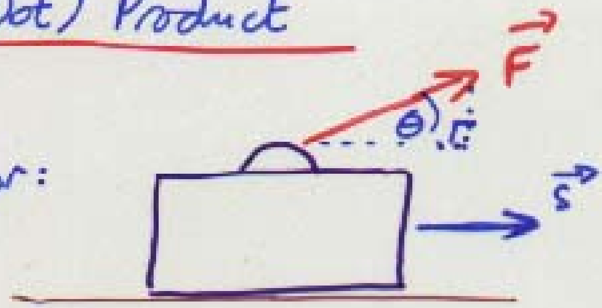
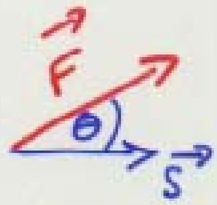
Work as a Vector (Dot) Product

e.g. Pulling suitcase along floor:



Applied force $\vec{F} = F\cos\theta\,\vec{i} + F\sin\theta\,\vec{j}$.

Suitcase displaced by $\vec{s} = l\,\vec{i}$



Work done $W = F\cos\theta \, . \, l$

The vector dot product $\vec{F}.\vec{s} = Fl\cos\theta$

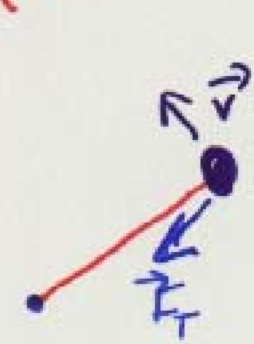
so work $W = \vec{F}.\vec{s}$: component of $\vec{F}$ $\perp$ to motion does no work.

Note: If force $\vec{F}$ $\perp$ to motion, $\vec{F}.\vec{s} = 0$ and work done $= 0$

e.g.

force $\vec{F}$ changes motion but does no work

e.g.



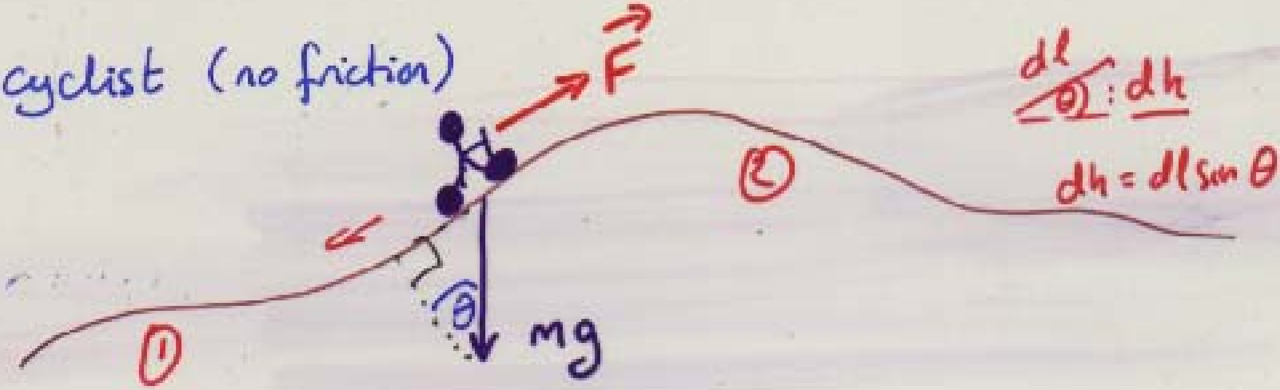
stone moves in circle on string

$\vec{F}_T$ changes $\vec{v}$ but

F_T $\perp$ motion $\Rightarrow$ no work done

Applications of Work: Gravity

e.g. For cyclist (no friction)



Weight $m\vec{g}$ acts downwards, cyclist must provide $F = mg\sin\theta$

over small distance dl, work $dW = F.dl = mg\sin\theta \, . \, dl$

i.e. $\int_1^2 dW = \int_1^2 mg \, \underline{dh}$. Integrate

$$\Rightarrow \quad W_{12} = mg\,(h_1 - h_2) = \underline{mg\,\Delta h}$$

- depends on height gain / drop only, <u>not</u> angle θ

As cyclist pedals uphill, h increases $\Rightarrow$ $W > 0$: work done by cyclist

As " freewheels downhill, h decreases $\Rightarrow$ $W < 0$: work done <u>on</u> cyclist

Note: work done $\propto$ change of height Δh
- independent of origin ($h = 0$)

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e.g. For $mg = 600\,N$, $\Delta h = 15\,m$ climb

$W = mg\,\Delta h = 600 \times 15 = 9\,kJ$. $\quad$ 1 Calorie $= 4.2\,kJ$

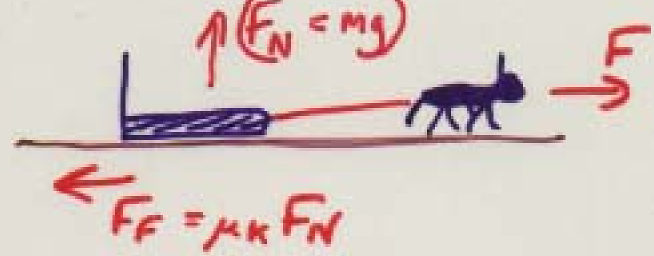
so $W = \dfrac{9.0}{4.2} \simeq 2$ Calories

Application of Work: Friction

e.g. Pull sled across flat ice-field at constant speed:

$v = \text{const} \Rightarrow a = 0$

so $F = F_F = \mu_K F_N$

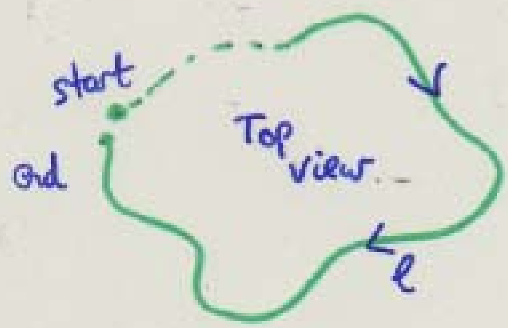


Dog does work on sled and ground (both heat up)

Net work over distance ℓ is (with $F_F \parallel$ to motion)
$\cos\theta = 0$

$$W = F_F \ell$$
$$= \mu m g \ell$$

Note: Path length ℓ = "odometer reading", always increases so $W > 0$ even for round-trip (cf gravity $\Delta h = 0 \Rightarrow W = 0$)



e.g. $\ell = 10\,\text{km}$, $mg = 2000\,\text{N}$, $\mu = 0.1$

$$\Rightarrow W = \mu m g \ell = 0.1 \times 2000 \times 10^4 m$$
$$= 2 \times 10^6\,\text{J}$$

Since $1\,\text{Cal} = 4.2\,\text{kJ}$

dog does $W = \dfrac{2 \times 10^6}{4.2 \times 10^3} \approx$ 480 Calories.

Work as an Integral : changing forces

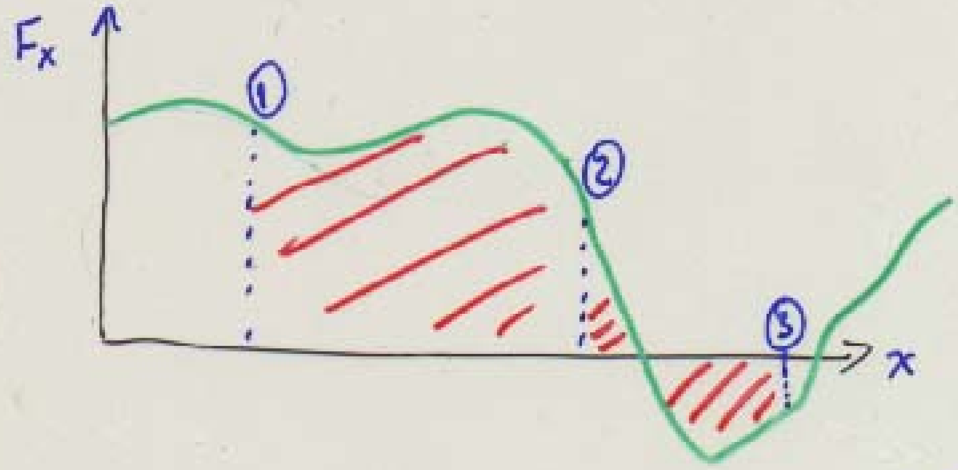
If force $\vec{F}$ changes along path:

work done by $\vec{F}$ over small disp. $d\vec{s}$ is

$$dW = \vec{F}.d\vec{s} = F ds \cos\theta$$

$\therefore$ Integrating along path : $W = \int \vec{F}.d\vec{s}$

e.g. For motion in 1-D with $|d\vec{s}| = dx$:



Work done between 1 and 2 $\quad W_{12} = \int_1^2 F_x \, dx$

also $W_{13} = W_{12} + W_{23}$ etc.

Work and Kinetic Energy (Work "stored" as motion)

To change speed of rigid body with a force requires WORK (force must push over distance).

Release object $\Rightarrow$ continues at constant $\vec{v}$ (Newton I).

BUT object can now transfer work by stopping

e.g. Play catch:

$\vec{F}_{HB} = m\vec{a}_1$ $\qquad\qquad\qquad$ $F_{BH} = m\vec{a}_2$

In flight, work is stored in ball's motion, released when brought to rest.

For constant force acting over distance x:

$W = F.x = m a x$ (Newton II)

Since $\frac{1}{2}v^2 = v_0^2 + 2\overset{F/m}{a}x$, $\underline{W = max = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}$

If choose $v_0 = 0$ $\Rightarrow$ $\underline{W = \frac{1}{2}mv^2}$

Stored work $\equiv$ Kinetic Energy $\underline{KE = \frac{1}{2}mv^2}$

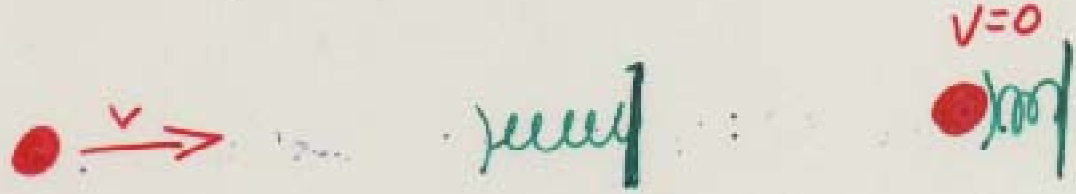
note: absolute value depends on reference frame $v=0$

but work W = change in KE for all observers

Potential Energy : Work stored by Position

e.g. Throw ball horizontally with speed v:

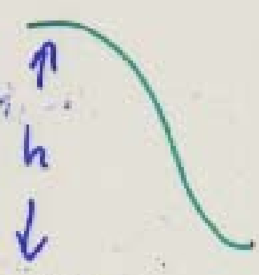
OR



Ball does work $W = \frac{1}{2}mv^2$ (against gravity or spring)

$\Rightarrow$ K.E. stored as Potential Energy (PE)

Later on:



Ball rolls downhill
gravity does work $F_W h = mgh$
on ball
$\rightarrow$ kinetic energy again.

OR compressed spring expands $\rightarrow$ restores ball's K.E.

In each case, KE is stored in the "configuration".

So Gravity: Potential Energy $= mgh$

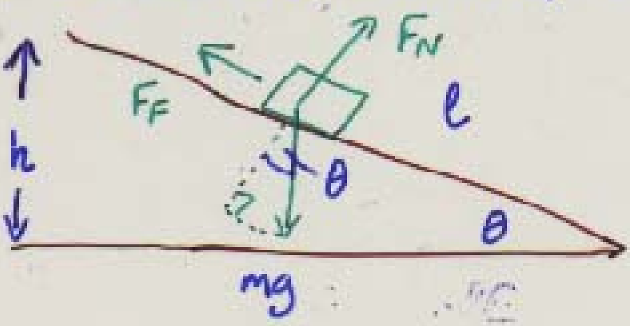
(depends on origin of $h=0$, but $\Delta PE = mg\Delta h$)

Friction: work dissipated as heat, sound etc.

$\rightarrow$ no energy stored (can't get it back!)

Problem Solving using "Energy Arguments"

e.g. mass slides down a ramp with friction Find final speed



$F_N = mg\cos\theta$

$\Rightarrow F_F = \mu F_N = \mu mg\cos\theta$

$h = l\sin\theta$

1. "Force method": net downhill force $= mg\sin\theta - \mu F_N$

   i.e. along plane: $m\ a = mg(\sin\theta - \mu\cos\theta)$

   $\therefore$ final speed $v^2 = \underbrace{v_0^2}_{=0} + 2al$

   $v^2 = 2mgl(\sin\theta - \mu\cos\theta)$ *

2. "Energy method": P.E. lost = KE gained + work done against friction

   i.e. $mgh = \frac{1}{2}m(v^2 - v_0^2) + F_F . l$

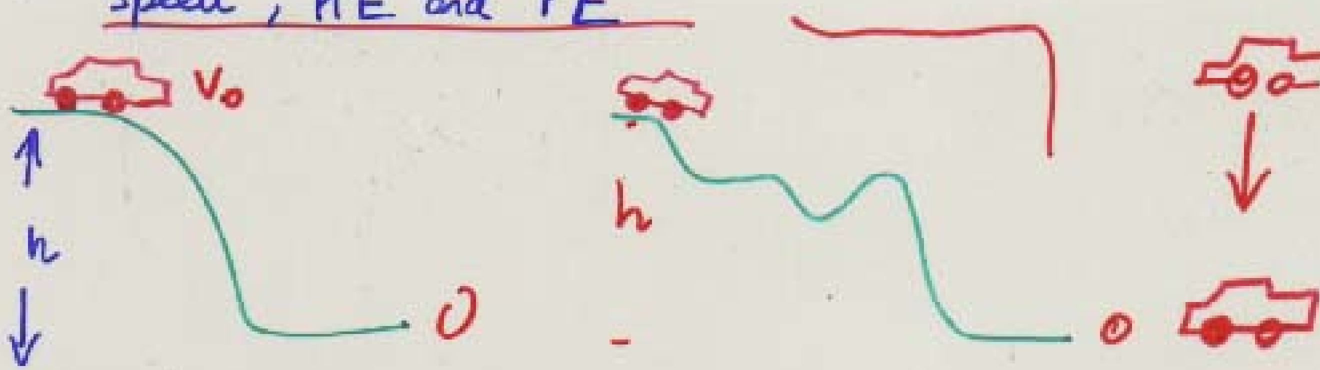
   with $h = l\sin\theta$, $F_F = \mu mg\cos\theta$

   $\Rightarrow \underbrace{\frac{1}{2}mv^2}_{KE_f} = mgh - F_F l$

   $= mgl\sin\theta - \mu mgl\cos\theta$ *

   $\Rightarrow$ same result as before.

Speed, KE and PE



If no friction, $\Delta KE = \Delta PE = mgh$ - depends on height change only

e.g. if car starts from rest $v_0 = 0$

$$\Delta KE = \tfrac{1}{2}mv^2 - 0 = mgh \Rightarrow v^2 = 2gh$$ (same as free-fall speed)

Note: Final v may be the same, but time taken depends on path.

e.g. for a $h = 5m$ descent, $v^2 = 2gh = 2 \times 10 \times 5 \Rightarrow v = 10 m/s$

Careful! What if we give car initial push $v_0 = 3 m/s$?

Is final speed now $10 + 3 = 13 m/s$? ✗

As before $\Delta KE = \tfrac{1}{2}m(v^2 - v_0^2) = mgh = 50m$ Joules

so $$v^2 = v_0^2 + 2gh = 3^2 + 100$$

$$\Rightarrow v = \sqrt{109} = 10.44 m/s$$

- speed is not energy.

Add friction? Then $\Delta KE = \tfrac{1}{2}m(v^2 - v_0^2) = mgh - \int F_F . dl$

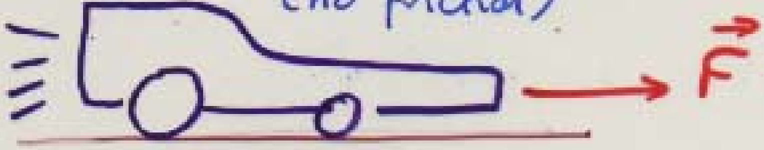
- reduces speed at bottom.

Example: Work to accelerate 500 kg car

(a) from 0 to 25 m/s

(b) from 25 to 50 m/s

(no friction)



a) $W = \Delta KE = \frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}.500 \times 25^2$

$= F_{av}.x$ $\quad = \underline{156\ kJ}$

b) $\Delta KE = \frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}.500(50^2 - 25^2)$

$= \underline{468\,kJ}$, i.e. 3x the work

for same change in speed

$\therefore$ for same force F_{av}, car must travel 3x the distance

though time interval $\Delta t = \dfrac{v - v_0}{(F/m)}$ is same for both

i.e. $\Delta t = \dfrac{\Delta v}{a} = \dfrac{m\Delta v}{F}$ (Newton II)

Note: rockets provide constant $|\vec{F}|$

but most engines $\rightarrow$ constant power $\dfrac{dW}{dt}$

Power = Rate of doing Work

Define : Power $P = \frac{\text{Work done}}{\text{time taken}}$ [J/s or Watt (W)]

e.g. If one engine raises a 100N (=mg) water bucket up a well with h = 20m in 30s

$$\frac{\text{Work}}{\text{time}} = \frac{mgh}{t} = \frac{100 \times 20}{30} = 66.7\ W$$

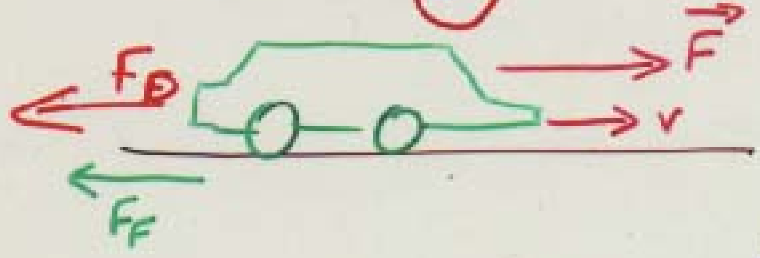
c.f. another motor does same in ½ the time

⇒ Power is doubled $\frac{mgh}{t} = 133.4\ W$ (≈ ⅓ hp garage door motor)

Power measures rate of energy transfer from force provider to point-of-application.

So if a force F pushes an object along at speed v :

$$\text{Power } P = \frac{\text{Work}}{\Delta t} = \frac{F\,\Delta \ell}{\Delta t} = F.v \quad \text{(force × speed)}$$



$P = Fv$

At top speed, $F = F_D$

$\therefore V_{max} = \frac{P}{F_D}$ for engine power P.