SHM Properties : Summary

Displacement: $x = A\cos(\omega t + \phi)$: ω depends on "physics"

A, ϕ set at $t=0$. e.g. $\phi = 0 \Rightarrow x(0) = A,\ v(0) = 0$

$\phi = -\frac{\pi}{2} \Rightarrow x(0) = 0,\ v(0) = -\omega A \neq 0$ $\quad x = A\sin\omega t$

Note: Period $T = 2\pi/\omega$: independent of amplitude !

Speed $v = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$

max. value $= \pm A\omega$ at $x = 0$

$= 0$ at $x = \pm A$ (reversing direction)

Accel $a = \frac{dv}{dt} = -\omega^2 x$ (and force = mass x accel)

$= 0$ at $x = 0$

$= \pm\omega^2 A$ at $x = \pm A$ (and when $v=0$)

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

$F \propto -x$

$$\boxed{\frac{md^2x}{dt^2} = -kx} \Longrightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x^2$$

$$= -\omega^2 x$$

KE and P.E. in SHM

$$x = A\cos(\omega t + \varphi)\ ;\ \omega^2 = k/m$$

$$\text{P.E. of spring} = \int_0^x F.dx = \tfrac{1}{2} k x^2 = \tfrac{1}{2} kA^2 \cos^2(\omega t + \varphi)$$

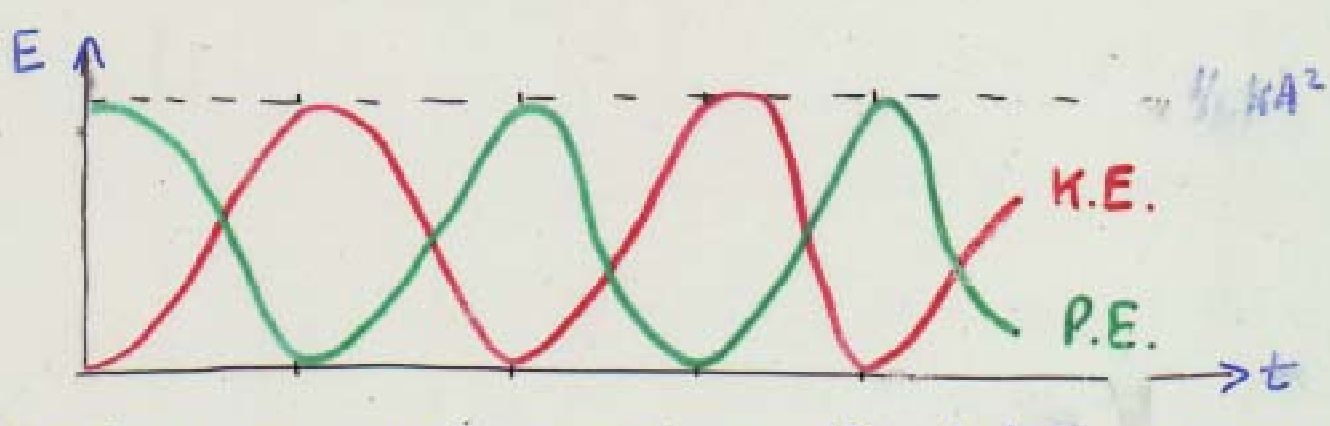
$$\text{K.E. of mass} = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \varphi)$$

But we know $\omega^2 = k/m$ from above

$$\Rightarrow KE = \tfrac{1}{2} \cancel{m} \cdot \frac{k}{\cancel{m}} A^2 \sin^2(\omega t + \varphi)$$

Total energy $\quad KE + PE = \tfrac{1}{2} kA^2 \Big[\underbrace{\sin^2(\cdots) + \cos^2(\cdots)}_{=1} \Big]$

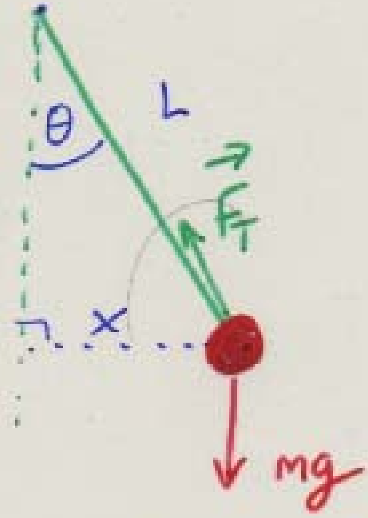
$E = \tfrac{1}{2} kA^2$, constant!



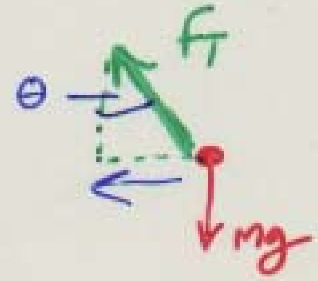
At ends of motion, $v = 0$, $x = \pm A$ and $PE = \tfrac{1}{2} kA^2$, $KE = 0$

At $x = 0$, $PE = 0$ and $KE = \tfrac{1}{2} m v_{max}^2 = \tfrac{1}{2} kA^2$.

Mass on a light String: Simple Pendulum



String length L



Displace mass by x from vertical; $\sin\theta = x/L$

Vertical forces: $F_T \cos\theta = mg$. If θ small, $\cos\theta = 1$

$F_T \simeq mg$.

Horizontal: net restoring force $= m\dfrac{d^2x}{dt^2} = -F_T \sin\theta$

Subs. $F_T \simeq mg$ and $\sin\theta = x/L \Rightarrow m\dfrac{d^2x}{dt^2} = -mg\dfrac{x}{L}$

i.e. $\dfrac{d^2x}{dt^2} = \dfrac{-g}{L}\,x$ c.f. $\dfrac{d^2x}{dt^2} = -\omega^2 x$ generally

$= 2\pi f$.

$\Rightarrow$ SHM with ang. speed $\omega = \sqrt{\dfrac{g}{L}}$, or

Period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$

Pendulum Properties: $T = 2\pi\sqrt{\frac{L}{g}}$

- Period T depends on $L, \vec{g}$ only <u>NOT mass</u> (c.f. spring $T = 2\pi\sqrt{\frac{k}{m}}$)

  - so child, adult on swing have same T

- As for all SHM, T independent of amplitude

  - pendulum keeps good time as amplitude ↓

- Can measure L, T to estimate "g" on earth and other planets

Example: For clock with $T = 2{\cdot}0\,s$ (1 "tick" = 1s):

required length $L = \frac{gT^2}{4\pi^2} = \frac{9.8 \cancel{\times 4}}{\cancel{4}\pi^2} = 0.993\,m$ on earth

Bio-mechanics:

Model (leg + foot) as pendulum for humans, animals, E.T. ....

⇒ "natural" period T of leg . c.f. giraffe, adult, child, dinosaurs

$T \propto \sqrt{L}$

On moon, Mars etc. $T \propto \frac{1}{\sqrt{g}}$: affects astronauts, other life

For adults on earth, natural "gait" or # steps/s $\propto f = \frac{1}{T} \propto \sqrt{\frac{g}{L}}$

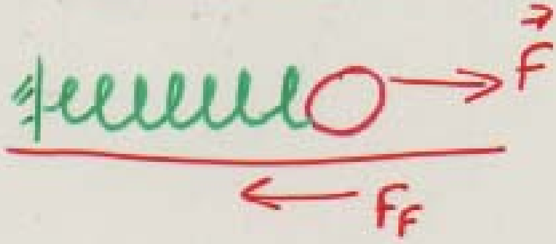
$\simeq 2$ step/s

Note: stride length $\ell \propto$ leg length L, so walking speed

$v = \text{stride} \times \text{gait} \propto L\sqrt{\frac{g}{L}} \propto \sqrt{gL}$ $\simeq 100$ yd/min on earth

– less on Moon, Mars . . .

Damped Vibrations: Energy Loss



Mass on spring sliding on table

In reality, total $E = KE + PE$ ↓ with time due to friction

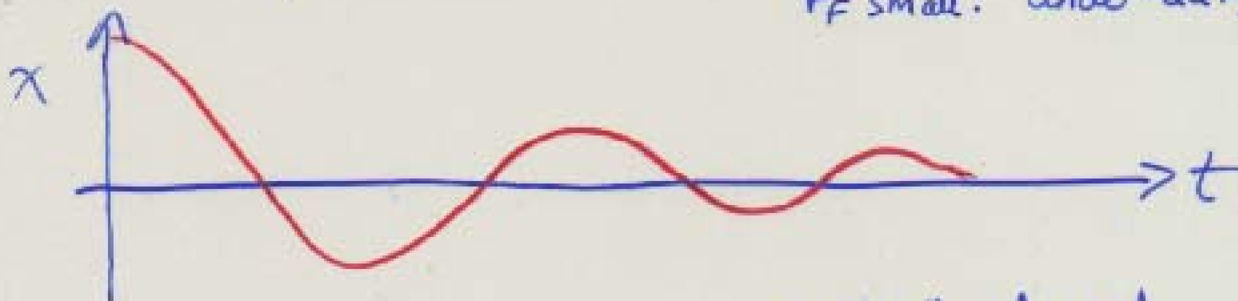
Since $E = \frac{1}{2}kA^2$, amplitude A ↓ also

energy lost = work done against friction at rate $\frac{dW}{dt} = F_F . v$

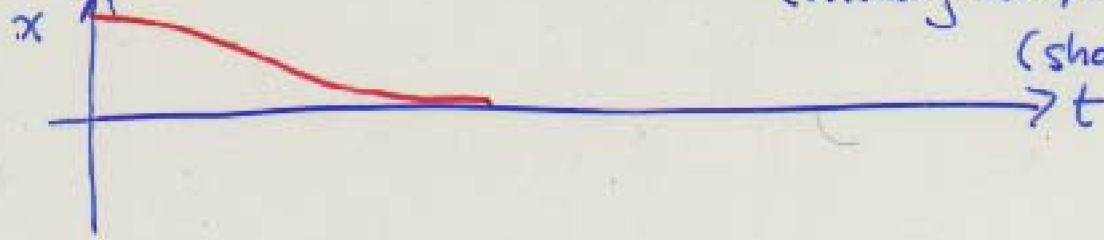
Can solve new eqn. of motion $\frac{md^2x}{dt^2} = -kx \mp F_F \frac{\vec{v}}{|\vec{v}|}$

3 classes of solutions:

F_F small: under-damped



critically damped (shock absorber)



over-damped : F_F large

Forced Oscillations : Resonance

Apply additional force to oscillator

$$m\frac{d^2x}{dt^2} = -kx + F(t)$$

Often $F(t)$ is periodic with driving frequency f_D

(e.g. road joints on freeway, push a swing, sing a note at wine glass, troops marching across bridge)

System absorbs energy from driving force when f_D is "in step" with natural frequency f

$\Rightarrow$ RESONANCE : every cycle absorbs energy

As $E\uparrow$, since $E = \frac{1}{2}kA^2$, $A\uparrow$ until

energy input rate = energy loss rate due to friction