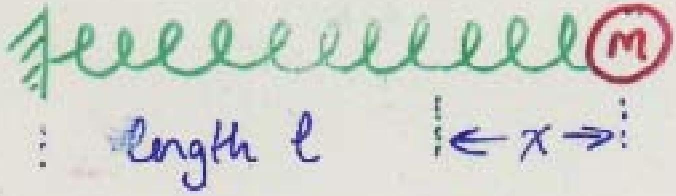
Vibrations of mass on a spring (ch. 10)



When mass displaced by x, restoring force $F = -kx$
(Hooke)

From Newton II: $F = m\dfrac{d^2x}{dt^2} = -kx$

or accel. $\dfrac{d^2x}{dt^2} = -\left(\dfrac{k}{m}\right) \cdot x \quad ; x(t)$

(accel. $\propto^{(-)}$ displacement)

Solution: $x = A\cos(\omega t + \varphi)$

Speed $v = \dfrac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$

Accel $a = \dfrac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \varphi) = -\boxed{\omega^2}x$

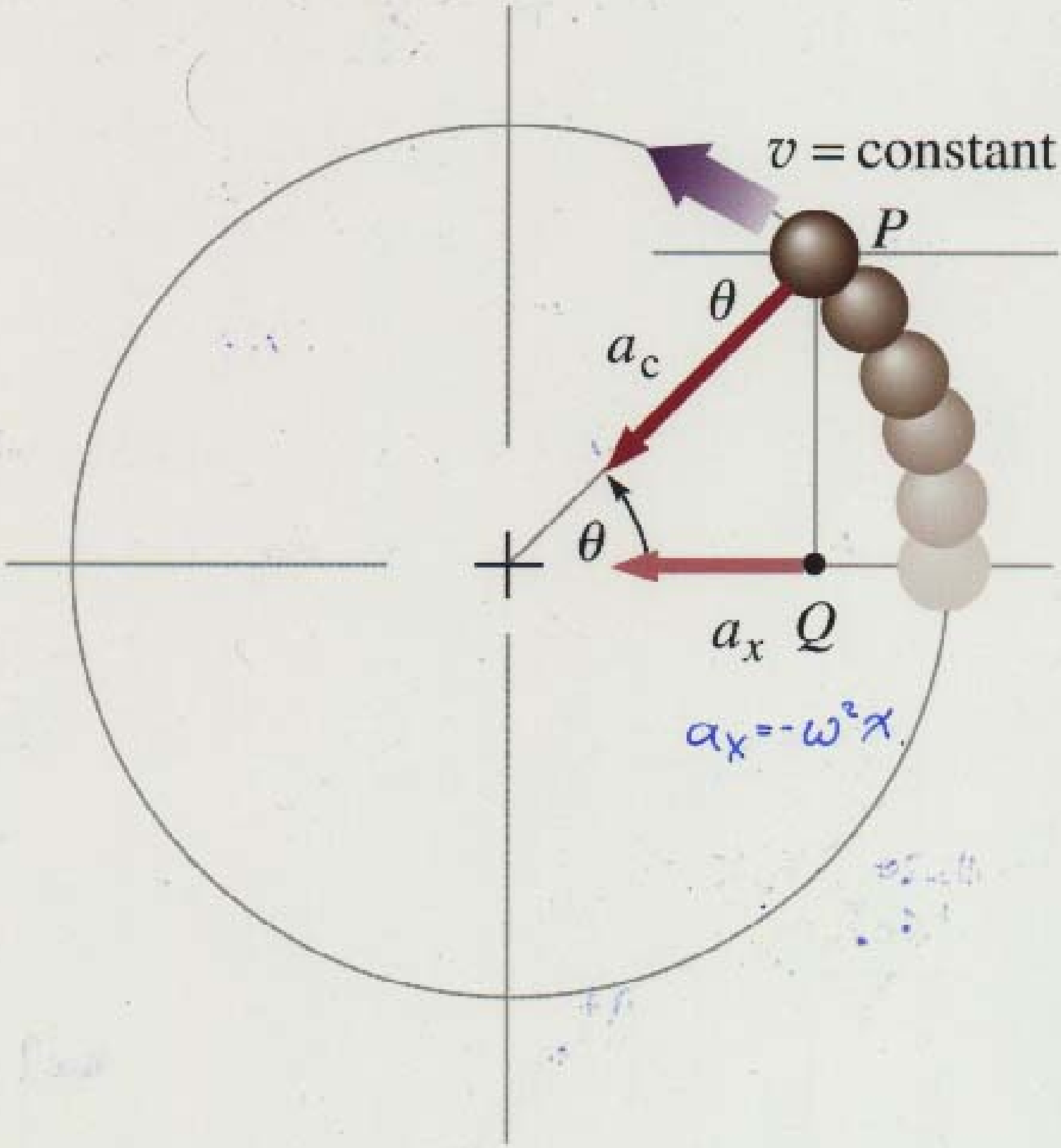
Solution works if $\omega^2 = \dfrac{k}{m}$ $[s^{-2}]$ ..

and constants A [m], φ [rad] set by

initial conditions (e.g. at $t=0$:

disp. $x = A\cos\varphi$ (1)
speed $v = -A\omega\sin\varphi$) (2)

Figure 10.26
Acceleration of an oscillator



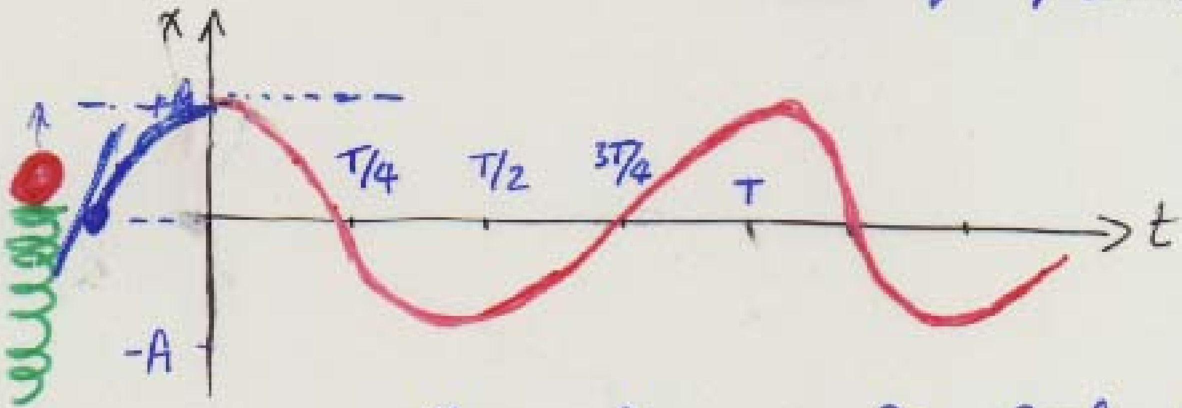
Simple Harmonic Motion (SHM)

Many systems have a restoring force $\propto$ $(-)$ displacement

e.g. child on swing, swaying trees, diatomic molecules

i.e. $F = m\dfrac{d^2x}{dt^2} = -kx$ or $\dfrac{d^2x}{dt^2} = -\omega^2 x$

$\Rightarrow$ Solution $x = A\cos(\omega t + \phi)$: harmonic vibration about eqm. position $(x=0)$.....



... with ang freq. $\omega = \dfrac{2\pi}{T} = 2\pi f$ [rad/s]

and initial phase angle ϕ.

If $\phi = 0$ (shown here), $x = +A$ at $t=0$

also $v = \dfrac{dx}{dt} = 0$ at $t=0$

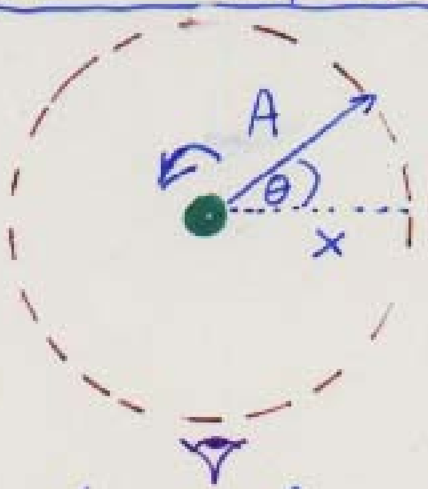
(e.g. displace object to $x = A$, then let go)

If $\phi = -\pi/2$, $x = A\cos(\omega t - \pi/2) = A\sin\omega t$

Then $x = 0$ at $t = 0$

$v = \dfrac{dx}{dt} = +A\omega$ at $t=0$ $(\omega > 0)$ (give object a "kick") impulse

SHM Properties



Start with uniform circular motion

radius $\equiv A$ frequency f [Hz]

angular frequency $\omega = 2\pi f$ [rad s^{-1}]

i.e. $\omega \equiv \dfrac{d\theta}{dt}$

So if $\theta = \varepsilon$ at $t = 0$

$\theta = \omega t + \varepsilon$ at time t

Projection $x = A \cos(\underbrace{\omega t + \varepsilon}_{\text{PHASE}})$

displacement

AMPLITUDE

SHM : Speed + Acceleration

$$x = A\cos(\omega t + \phi)$$

Speed $v = -A\omega\sin(\omega t + \phi) = A.\frac{2\pi}{T}\sin(\cdots)$

- depends on amplitude and frequency

Note, using $\sin^2\theta + \cos^2\theta = 1$ to eliminate "t"

$$\Rightarrow \underline{v = \pm A\omega\sqrt{1 - (x/A)^2}}$$

- has value $\pm A\omega$ at $x = 0$

$= 0$ at $x = \pm A$ (extremes of motion)

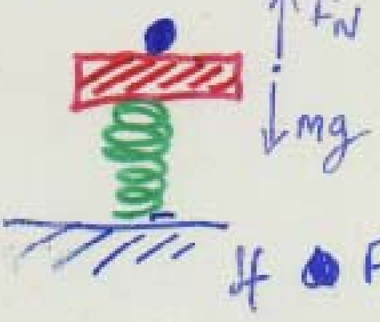
e.g. take photo of child on swing at ends of motion, not at "bottom" ($x=0$) when $v \neq 0$.

Acceleration $a = \frac{dv}{dt} = -\omega^2 A\cos(\cdots) = -\omega^2 x$

$= 0$ at $x=0$ (so object keeps moving)

$= \pm\omega^2 A$ at $x = \pm A$ when $v = 0$

e.g. for flea on mass on vertical spring, effective weight F_N

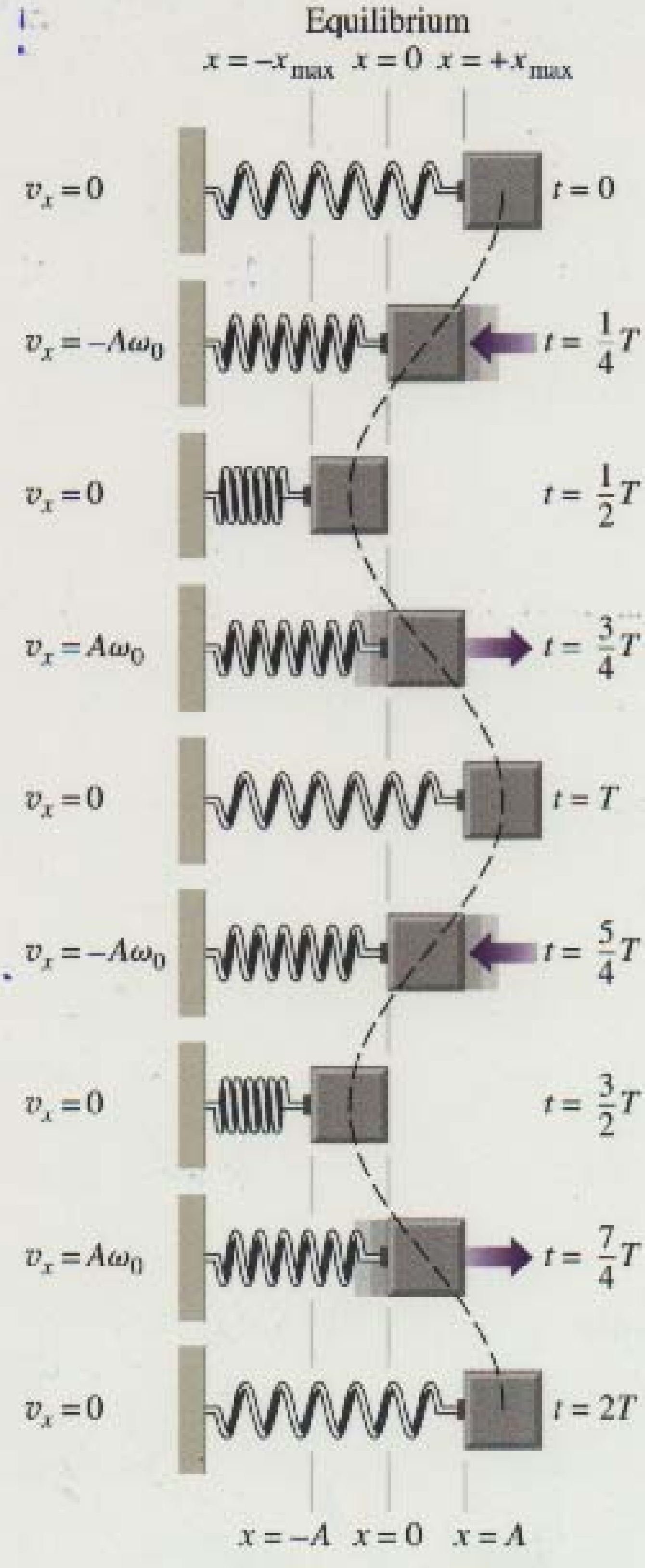


given by

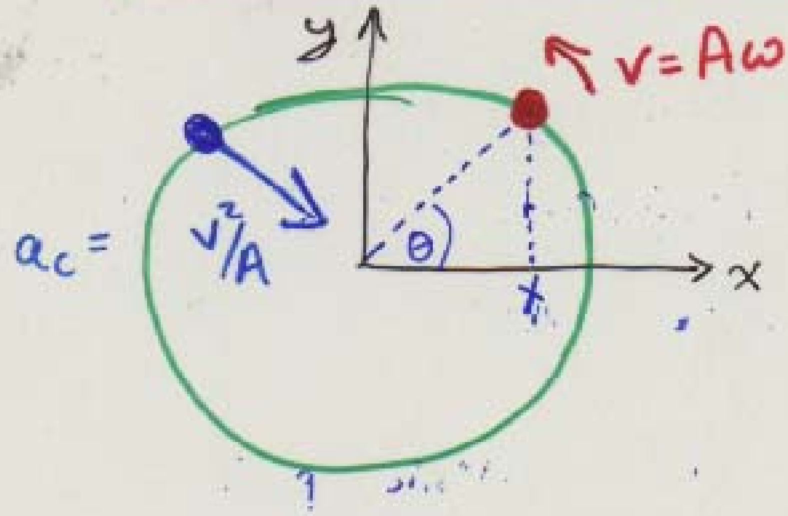
$$F_N = mg \pm ma = mg \pm \omega^2 x$$

If $F_N \leq 0$, platform "leaves flea behind".

Figure 10.28
Simple harmonic oscillator



Relation to Circular Motion



Object moves in circle, radius = A at uniform ang. speed $\omega = \frac{d\theta}{dt}$ [rad s^{-1}]


: View edge-on (e.g. Jupiter's moons)

At time t, angle $\theta = \omega t + \varphi$ ($= \varphi$ at $t=0$)

Observer sees only x-coordinate $x = A\cos\theta$

i.e. $x = A\cos(\omega t + \varphi)$ *

Also, x-component of centripetal accel $a_c = \frac{v^2}{A}$ is

$$a_x = a_c\cos\theta = \frac{v^2}{A}\cdot\frac{x}{A} = \frac{v^2}{A^2}\cdot x = (-)\underline{\omega^2 x} \text{ as before}$$

Period of cycle $T = \frac{2\pi}{\omega}$. Frequency $f = 1/T = \frac{\omega}{2\pi}$ (s^{-1} or Hertz).

Max. displacement along x-axis is $x = \pm A$ [m]
(when $\cos(\omega t + \varphi) = \pm 1$)

A : amplitude of oscillation

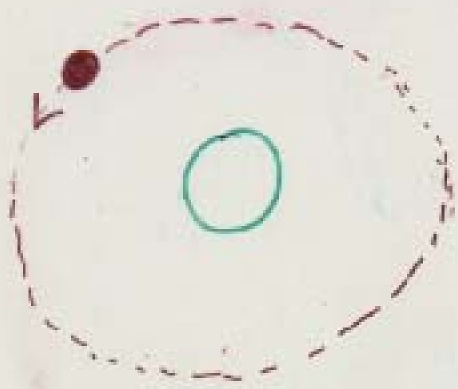
φ : Initial phase angle [rad] at $t=0$.

SHM Examples:

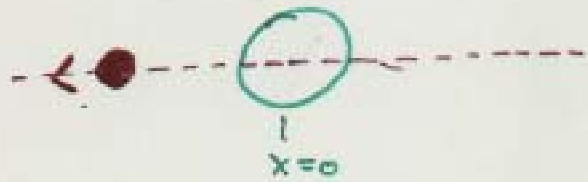
- Pendulum - child on swing

- Mass on spring
- Diatomic molecules

- "Edge-on" circular orbit (Fig 12.8)



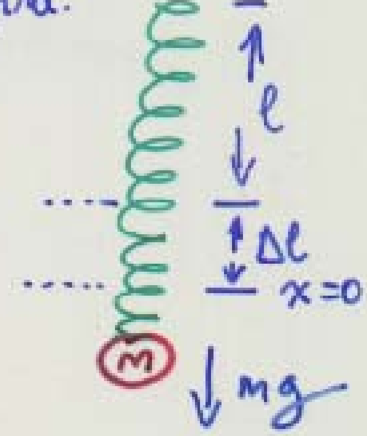
Top View



Side view

Example: Mass on Vertical Spring

A spring has un-extended length 0.85m. A 500g mass is attached, then let go. If $k = 10\ N/m$; find:



1. New eqm. length of spring
2. Amplitude + period of oscillation
3. Max. speed and accel of mass

1. New eqm. length is where $k\Delta l = mg$ i.e. $\Delta l = \frac{mg}{k} = \frac{10 \times 0.5}{10}$

$\Rightarrow$ new length $\underline{l + \Delta l} = 0.85 + 0.5 = 1.35m$.

2. Set $x = 0$ at this length. For displacement x (down)

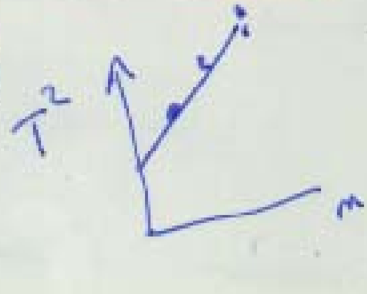
$$\text{Net force } F = m\frac{d^2x}{dt^2} = \cancel{mg} - \cancel{k}(\underbrace{\Delta l + x})$$

Now $mg = k\Delta l$ so $\Rightarrow$ $m\frac{d^2x}{dt^2} = -kx$, oscillates about $x = 0$.

Ang. freq $\omega = \sqrt{\frac{k}{m}} = 4.47$ rad/s. Period $T = 2\pi/\omega$

i.e. $T = 2\pi\sqrt{\frac{m}{k}} = 1.41s$. Note $T^2 \propto m$ (lab)

in fact we find $T^2 \propto (m + \frac{1}{3}M_s)$



eg. A mass of 0.4 kg vibrates on a spring with $k = 100\,\text{kg/s}^2$ (N/m)

Mass is given an initial "kick" of 0.5 m/s.

Find the amplitude of the motion, and the eqn. of motion.

$\frac{1}{2} \times 0.4 \times (0.5\,\text{m/s})^2$

1. Initial $KE = \frac{1}{2}mv^2 = 0.05\,J$, initial $PE = 0$

So max. PE (when $x = \pm A$ and $v = 0$) is $\frac{1}{2}kA^2 = 0.05\,J$

$\Rightarrow \quad A = 0.0316\,m.$

2. $\quad x = A\cos(\omega t + \varphi)$. $\quad \omega = \sqrt{\frac{k}{m}} = 15.8\,\text{rad/s}$

Speed $v = -\omega A \sin(\omega t + \varphi) = 0.5\,\text{m/s}$ at $t = 0$

So at $t=0$:

$$0.5\,\text{m/s} = -\omega A \sin\varphi \quad (1)$$

$$x = 0 = A\cos\varphi \quad (2)$$

(1), (2) only satisfied if $\varphi = -\pi/2$ $(-90°)$

Then $\sin\varphi = -1$ and we can check $v(0) = \omega A = 15.8 \times 0.316 = 0.5\,\text{m/s}$ ✓✓

So motion is $x(t) = 0.316\,m \cos(15.8t - \pi/2)$

i.e. $\quad x = 0.316\,m \sin 15.8t$